ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of
APPLIED
MATHEMATICS
AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 69 (2005) 792–804

CONTACT INTERACTION OF A PIEZOELECTRIC ACTUATOR AND ELASTIC HALF-SPACE†

I. D. KOCHETKOV and N. N. ROGACHEVA

Moscow

(*Received* 23 *November* 2004)

An analytical solution is obtained of the following contact problem: a piezoelectric actuator in the form of a thin infinite strip, placed on the surface of an elastic half-space, performs oscillations due to the action of an electric load, which excites surface and bulk waves. The behaviour of these waves at all points of the elastic body, and also the stress–strain state of the piezoelectric actuator, are investigated. As an example, a numerical calculation is made of the displacements of the acoustic waves of an elastic half-space of steel, excited by a piezoelectric actuator made of PZT-5 piezoelectric ceramics, and the stress–strain state of the piezoelectric actuator is also calculated. It is shown that the amplitudes of the surface and bulk waves depend very much on the oscillation frequency. The attenuation of acoustic waves with distance from the oscillation source both in the depth of the body and on its surface is investigated.

In an isotropic elastic medium having a boundary, surface waves can propagate in addition to longitudinal and transverse (shear) waves [1–5]. The displacement of the surface wave, which satisfies the conditions that there should be no mechanical stresses on the free surface of the body, contains both longitudinal and transverse components. The surface waves propagate over the surface without attenuation. They penetrate only a small depth into the body as a consequence of the rapid attenuation.

In the majority of publications devoted to a acoustic waves (see, for example, [6–8]), a solution of the homogeneous problem is constructed, just like Rayleigh did. As a result only the velocity, wavelength and wave numbers are found, but the amplitude values of the required quantities are not determined. The determination of the amplitude values is a complex problem even in the simple case, for example, when a concentrated force acts on the surface of the elastic half-space [2, 9].

1. FORMATION OF THE PROBLEM

We will solve the following contact problem: a piezoelectric actuator in the form of a thin infinite strip is placed on the surface of an elastic half-space. The actuator executes oscillations due to the action of an electric load, which excites surface and body waves in the elastic medium.

In order to simplify the complex contact problem, we will make certain obvious assumptions regarding the electroelastic state of the piezoelectric actuator.

The complete problem, taking into account the assumptions made, can be separated into the problem for the piezoeletric actuator and the sum of the auxiliary problems for the elastic half-space. The solution of the simplified system of equations for the piezoelectric actuator is sought in the form of Fourier series. To solve the auxiliary problems for the elastic half-space an integral Fourier transformation is employed. The integrals thereby obtained are reduced to a form convenient for calculations by the method of functions of a complex variable. The complete solution of the elastic half-space is written as the sum of the solutions of the auxiliary problems. The solutions of the problems for the piezoelectric actuator and the elastic half-space contain unknown constants, which ensure that the contact conditions are satisfied. It should be noted that the solution obtained enables one to determine all the required quantities of the contact problem, including their amplitude values.

†*Prikl. Mat. Mekh.* Vol. 69, No. 5, pp. 882–895, 2005.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.09.014

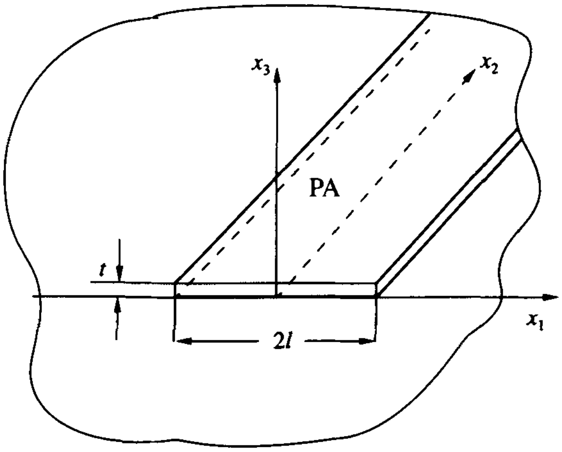

Fig. 1

2. THE PROBLEM FOR A PIEZOELECTRIC ACTUATOR

In Fig. 1 we show a piezoelectric actuator (PA) in the form of a thin infinite strip and an elastic half-space in Cartesian system of coordinates. It is assumed that the piezoelectric actuator, made of piezoelectric ceramics polarized in the thickness direction, has a width $2l$, a thickness t and is infinite in the x_2 direction. The piezoelectric actuator performs harmonic oscillations due to the action of an electric load, applied to the electrodes, which cover the end surfaces of the piezoelectric actuator $x_3 = 0$ and $x_3 = t$. The electric load varies harmonically with an angular frequency of oscillation ω, and hence all the equations can be written in terms of the amplitude values of the required quantities. The solution of the contact problem is independent of the x_2 coordinate, since the piezoelectric actuator is infinite in the x_2 direction.

The system of equations describing the piezoeletric actuator in terms of amplitude values consists of the following [10, 11] (below we only give those equations which will be used to solve the contact problem formulated above)

the equilibrium equations

$$\sigma_{1i,1} + \sigma_{i3,3} + \rho_p\omega^2 u_i = 0, \quad i = 1, 3 \tag{2.1}$$

the equations of state

$$\sigma_{kk} = \frac{1}{s_{11}^E(1-\nu_p^2)}(e_k + \nu_p e_n) - \left\{\frac{s_{13}^E}{s_{11}^E(1-\nu_p)}\sigma_{33}\right\} - \frac{d_{31}}{s_{11}^E(1-\nu_p)}E_3, \quad \nu_p = -\frac{s_{12}^E}{s_{11}^E} \tag{2.2}$$

$$D_3 = \varepsilon_{33}^E E_3 + d_{31}(\sigma_{11} + \sigma_{22}) + \{d_{33}\sigma_{33}\} \tag{2.3}$$

and relations connecting the strains and the displacement

$$e_k = u_{k,k}, \quad e_2 = u_{2,2} = 0 \tag{2.4}$$

The component E_3 of the electric field vector is related to the electric potential ϕ by the formula

$$E_3 = -\phi_{,3} \tag{2.5}$$

Here $k \neq n$ and $k = 1, 2$.

The following electric potential is specified on the electrodes of the piezoelectric actuator

$$\phi|_{x_3=0} = -V, \quad \phi|_{x_3=t} = +V \tag{2.6}$$

In Eqs (2.1)–(2.5) σ_{11}, σ_{22}, σ_{13} and σ_{33} are the stresses, D_3 is the component of the electric induction vector in the x_3 direction, s_{11}^E, s_{12}^E and s_{13}^E are the elastic compliances for zero electric field, d_{31} and d_{33} are the piezoelectric constants and and ε_{33}^T is the permittivity for zero stresses.

We will solve the class of problems in which:

(1) the piezoelectric actuator is thin (its thickness t is considerably less than its width $2l$ ($t \ll 2l$)),

(2) the wavelength in the elastic medium is an order or more greater than the width of the piezoelectric actuator (this means that the variability of the stress–strain state of piezoelectric actuator along the x_1 coordinate is small).

For the case considered, the problem can be simplified as a result of the following hypotheses.

1. The stresses σ_{33} are considerably less than the stresses σ_{11} and σ_{22}, and hence we can neglect σ_{33} compared with σ_{11} and σ_{22} in Eqs (2.2)–(2.3).

2. The displacements u_1 and u_3 are constant along the thickness (the displacements change so slowly along the x_3 coordinate that we can assume that they are independent of x_3).

3. E_3 is independent of x_3.

If these hypotheses are not satisfied, this method of solving the problem is useless. For example, if the wavelength of the acoustic waves is of the same order or less than the thickness of the piezoelectric strip, the first hypothesis is not satisfied. If the surface of the elastic half-space is completely covered with a piezoelectric layer, which is an unbounded piezoelectric medium, in this special case the stresses σ_{33} are not small compared with σ_{11} and σ_{22}, and it is easy to obtain a simple analytical solution of the problem which is qualitatively different from that investigated here.

The above hypotheses are not new. The first and second hypotheses are generally employed, beginning with Kirchhoff, for any thin layer having small variability along the x_1 and x_2 coordinates. All the hypotheses are usually used for thin-walled piezoelectric strips and sensors. The correctness of the hypotheses is confirmed by numerous experiments [11, 12]. Moreover, the hypotheses have been confirmed by an asymptotic analysis [13].

Using the first and second hypotheses, in Eqs (2.2)–(2.3) we must neglect the terms enclosed in the braces.

It follows from the third hypothesis and Eq. (2.5) that the electric potential is a linear function of the thickness coordinate x_3, while E_3, taking condition (2.6) into account, can be expressed in terms of V as follows:

$$E_3 = -2V/t \tag{2.7}$$

Since u_1, e_1, and E_3 are independent of the variable x_3, it can be seen from Eqs (2.2)–(2.4) that σ_{11} and D_3 are also independent for x_3 and are functions solely for x_1.

Integration of Eqs (2.1) with respect to x_3 and satisfaction of the conditions for there to be no stresses on the surface $x_3 = t$

$$\sigma_{13} = 0, \quad \sigma_{33} = 0 \tag{2.8}$$

leads to the following formulae for the stresses σ_{13} and σ_{33}

$$\sigma_{13} = (\sigma_{11,1} + \rho_p\omega^2 u_1)(t - x_3) \tag{2.9}$$

$$\sigma_{33} = (\sigma_{11,11} + \rho_p\omega^2 u_{1,1})((t^2 + x_3^2)/2 - tx_3) + (t - x_3)\rho_p\omega^2 u_3 \tag{2.10}$$

In order to solve the problem in terms of Fourier series, we must extend the electric load E_3, as a periodic function of x_1 in the interval $[-L, +L]$ ($L > 2l$)

$$F(x_1) = \begin{cases} E_3, & |x_1| \le l \\ f(x_1), & l \le |x_1| \le L - l; \\ -E_3, & L - l \le |x_1| \le L \end{cases} \qquad \int_{-L}^{+L} F(x_1)dx = 0 \tag{2.11}$$

The function $f(x_1)$ is chosen so that the function $F(x_1)$ is continuous in the section $[-L, +L]$.

Note that function can be extended by any method which ensures rapid convergence of the solution.

The extended function E_3 must then be expanded in a Fourier series (everywhere henceforth summation over n is carried out from $n = 1$ to N)

$$E_3 = \sum_n \phi_n \cos a_n x_1, \quad \phi_n = \int_{-L}^{+L} F(x_1)\cos a_n x_1 dx_1, \quad a_n = \frac{n\pi}{L} \tag{2.12}$$

All the required quantities for the piezoelectric actuator will be sought in the form of trigonometric series

$$\begin{aligned} u_1 &= \sum_n u_{1n}\sin a_n x_1, \quad u_3 = \sum_n u_{3n}\cos a_n x_1 \\ \sigma_{11} &= \sum_n \sigma_{11n}\cos a_n x_1, \quad \sigma_{13} = \sum_n \sigma_{13n}\sin a_n x_1, \quad \sigma_{33} = \sum_n \sigma_{33n}\cos a_n x_1 \end{aligned} \tag{2.13}$$

3. THE PROBLEM FOR AN ELASTIC HALF-SPACE

The complete system of equations describing the elastic half-space $x_3 \le 0$, includes the equilibrium equations (2.1), formulae connecting the strains and displacements, and Hooke's law.

To solve the problem we will use the method proposed by Lamb for the case of concentrated forces, applied at a point of the surface of the elastic body [2, 9].

Following Lamb, we will write the equilibrium equations in displacements (everywhere henceforth $i = 1, 3$)

$$(\lambda + \mu)\theta_{,i} + \mu\nabla^2 u_i + \rho\omega^2 u_i = 0 \tag{3.1}$$

where

$$\theta = u_{1,1} + u_{3,3}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{3.2}$$

(θ is the relative volume deformation, and λ and μ are the Lamé coefficients).

Hooke's law can be written in the form

$$\sigma_{ii} = \lambda(e_1 + e_3) + 2\mu e_i, \quad \sigma_{31} = \mu e_{31}, \quad e_i = u_{i,i}, \quad e_{13} = u_{1,3} + u_{3,1} \tag{3.3}$$

The functions φ and ψ, related to the displacements, are introduced in the usual way

$$u_1 = \varphi_{,1} + \psi_{,3}, \quad u_3 = \varphi_{,3} - \psi_{,1} \tag{3.4}$$

The equilibrium equations and the formulae for the stresses, expressed in terms of the functions φ and ψ can be written as follows:

$$(\nabla^2 + h^2)\varphi = 0, \quad (\nabla^2 + k^2)\psi = 0 \tag{3.5}$$

$$\sigma_{11} = \mu(-k^2\varphi - 2\varphi_{,33} + 2\psi_{,13}) \tag{3.6}$$

$$\sigma_{33} = \mu(-k^2\varphi - 2\varphi_{,11} - 2\psi_{,13}), \quad \sigma_{31} = \mu(-k^2\psi - 2\psi_{,11} + 2\varphi_{,13}) \tag{3.7}$$

$$h^2 = \frac{\omega^2}{c_L^2}, \quad k^2 = \frac{\omega^2}{c_T^2}, \quad c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_T = \sqrt{\frac{\mu}{\rho}}$$

where c_L and c_T are velocities, and h and k are the wave numbers of the longitudinal and transverse waves respectively.

To solve the problem we will use a Fourier integral transformation with respect to the variable x_1

$$\varphi^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi e^{-i\xi x_1} dx_1, \quad \psi^* = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \psi e^{-i\xi x_1} dx_1 \tag{3.8}$$

Since the x_3 axis is directed from the elastic half-space (Fig. 1), the solution in the elastic half-space must decrease as x_3 decreases, and hence the solution is taken in the form

$$\varphi^* = Ae^{\alpha x_3}, \quad \psi^* = Be^{\beta x_3} \tag{3.9}$$

where α and β are positive real quantities, which ensures that the stress–strain state of the elastic body decreases with distance from the source of oscillations into the depth of the body.

In order to obtain the quantities α and β we must carry out the integral transformation (3.8) in Eqs (3.5) and substitute expressions (3.9) into them

$$\alpha = (\xi^2 - h^2)^{1/2}, \quad \beta = (\xi^2 - k^2)^{1/2} \tag{3.10}$$

As a result of using the Fourier transformation, Eqs (3.4) and (3.7) take the form

$$u_1^* = i\xi Ae^{\alpha x_3} + \beta Be^{\beta x_3}, \quad u_3^* = \alpha Ae^{\alpha x_3} - i\xi Be^{\beta x_3} \tag{3.11}$$

$$\sigma_{33}^* = \mu((2\xi^2 - k^2)Ae^{\alpha x_3} - 2i\xi\beta Be^{\beta x_3}), \quad \sigma_{13}^* = \mu((2\xi^2 - k^2)Be^{\beta x_3} - 2i\xi\alpha Ae^{\alpha x_3}) \tag{3.12}$$

The arbitrary constants A and B are found from the condition for the piezoelectric actuator and the elastic half-space to be in contact in the $x_3 = 0$ plane.

In Section 2 the electric load E_3 is extended as a periodic function of the variable x_1 into the interval $(-\infty, +\infty)$ (2.11) and is expanded in a Fourier series (2.12). The remaining required quantities of the piezoelectric actuator are also written in the form of Fourier series. The problem for an elastic half-space is solved using an integral Fourier transformation. For the integral transformation of the stresses σ_{33} and σ_{31} we take into account these equality of these stresses, corresponding to the stresses of the piezoelectric actuator (2.13) in the contact area, and the fact that they are zero on the remaining surface of the elastic body. A similar method is used in hydroacoustics to solve the problem of the oscillations of a hinged cylindrical shell of finite length, immersed in an infinite liquid [14].

In order to solve the contact problem, we will first solve two auxiliary problems. In the first auxiliary problem only a normal surfaces load $\sigma_{33} = \cos a_n x_1$ acts on the surface of the elastic half-space in the contact area with a piezoelectric actuator, while in the second problem only a shear load $\sigma_{31} = \sin a_n x_1$ acts in the contact area.

The first auxiliary problem for an elastic half-space. We will assume that the normal stresses are specified in the contact area on the surface of the elastic half-space, while the remaining part of the surface is stress-free

$$\begin{aligned} &\sigma_{33}|_{x_3=0} = X_3, \quad \sigma_{13}|_{x_3=0} = 0, \quad X_3 = \cos px_1, \quad p = a_n = n\pi/l, \quad |x_1| \le l \\ &\sigma_{33}|_{x_3=0} = 0, \quad \sigma_{13}|_{x_3=0} = 0, \quad |x_1| > l \end{aligned} \tag{3.13}$$

Conditions (3.13), as a result of the Fourier transformation,

$$\sigma_{33}^*|_{x_3=0} = X_3^*, \quad \sigma_{13}^*|_{x_3=0} = 0, \quad X_3^* = -\frac{i}{2\pi}(\Omega^-(\xi) + \Omega^-(-\xi)) \tag{3.14}$$

where

$$\Omega^{\pm}(\xi) = \exp(i(p+\xi)l/(p+\xi)) \pm \exp(-i(p-\xi)l/(p-\xi)) \tag{3.15}$$

and taking formulae (3.12) into account, can be rewritten in the form

$$(2\xi^2-k^2)A-2i\xi\beta B = \frac{X_3^*}{\mu}, \quad 2i\xi\alpha A+(2\xi^2-k^2)B = 0 \tag{3.16}$$

The constants A and B are defined as the solution of system (3.16), after which the displacements are found by applying an inverse Fourier transformation to expressions (3.11)

$$\begin{aligned} u_1 &= \frac{i}{\mu}\int_{\infty}^{+\infty}[(2\xi^2-k^2)e^{\alpha x_3}-2\alpha\beta e^{\beta x_3}]\frac{\xi X_3^*}{F(\xi)}e^{i\xi x_1}d\xi \\ u_3 &= \frac{i}{\mu}\int_{\infty}^{+\infty}[(2\xi^2-k^2)e^{\alpha x_3}-2\xi^2 e^{\beta x_3}]\frac{\alpha X_3^*}{F(\xi)}e^{i\xi x_1}d\xi \end{aligned} \tag{3.17}$$

The quantities α and β are found from formulae (3.10), where $F(\xi)$ in Rayleigh's function

$$F(\xi) = (2\xi^2-k^2)^2-4\xi^2\alpha\beta \tag{3.18}$$

Integrals (3.17) differ from the well-known solution for the case of a concentrated load, applied at a point of the surface of an elastic-space [9], by the factors X_3^* in the integrand. However, in formulae (3.17) we have taken into account the dependence on the x_3 coordinate (in [9] formulae were obtained for the displacements on the surface of an elastic body). As in [9], the integrals (3.17) are transformed by choosing an appropriate contour of integration in the complex plane $\zeta = \xi + i\eta$. The integrals (3.17), in exactly the same way as in [9], have two poles $\zeta = \pm\kappa$, which are the roots of the Rayleigh equation

$$F(\varsigma) = 0 \tag{3.19}$$

where $\kappa > k$, and, in exactly the same way as in [9], four branching points $\zeta = \pm h$, $\xi = \pm k$ due to the presence of the radicals $(\zeta^2-h^2)^{1/2}$ and $(\zeta^2-k^2)^{1/2}$ in expressions (3.18) and (3.10), and, moreover, two poles $\zeta = \pm p$ of the factor X_3^*. The transformation of the integrals (3.17) is omitted here since it is similar to the well known transformations in [9].

The area of contact of the piezoeletric strip with the elastic body $|x_1| \le l$ and the region outside the contact area $|x_1| > l$ must be considered separately.

As a result of the transformations, the integrals (3.17) can be reduced to the form

$$\begin{aligned} u_1^{(n)} &= \frac{i}{2\mu}H_1^{(n)}e^{-i\kappa x_1}+\frac{i}{2\mu}P_1^{(n)}e^{-ipx_1}+iI_{k1}^{(n)}+iI_{h2}^{(n)} \\ u_3^{(n)} &= -\frac{i}{2\mu}K_1^{(n)}e^{-i\kappa x_1}-\frac{i}{2\mu}Q_1^{(n)}e^{-ipx_1}-2I_{k3}^{(n)}-\frac{1}{2}I_{h4}^{(n)} \end{aligned} \tag{3.20}$$

Here

$$I_{gm}^{(n)} = \frac{1}{\pi\mu}e^{-igx_1}\int_0^{\infty}R_m^{(n)}e^{-\eta x_1}d\eta, \quad g = k, h, \quad m = 1, 2, 3, 4$$

$$\frac{H_1^{(n)}}{T_1(\alpha_\kappa, \beta_\kappa)} = \frac{K_1^{(n)}}{T_2(\alpha_\kappa, \beta_\kappa)} = \frac{Z_3(-\kappa)}{F'(\kappa)}, \quad \frac{P_1^n}{T_1(\alpha_p, \beta_p)} = \frac{Q_1^{(n)}}{T_2(\alpha_p, \beta_p)} = -\frac{1}{F(p)} \tag{3.21}$$

$$R_1^{(n)} = \gamma_1(\alpha, \beta)\frac{\varsigma\alpha\beta}{2F_2(\varsigma)}Z_3(\varsigma)\Big|_{\varsigma=-k+i\eta}, \quad R_2^{(n)} = \gamma_2(\alpha, \beta)\frac{\varsigma\Phi(\varsigma)}{4F_2(\varsigma)}Z_3(\varsigma)\Big|_{\varsigma=-h+i\eta}$$

$$R_3^{(n)} = -\gamma_2(\beta, \alpha)\frac{\varsigma\alpha\beta}{2F_2(\varsigma)}Z_3(\varsigma)\Big|_{\varsigma=-k+i\eta}, \quad R_4^{(n)} = \gamma_1(\beta, \alpha)\frac{\alpha\Phi(\varsigma)}{4F_2(\varsigma)}Z_3(\varsigma)\Big|_{\varsigma=-h+i\eta}$$

where

$$
\begin{aligned}
&T_1(\alpha_g, \beta_g) = g[\Phi(g)e^{\alpha_g x_3} - 2\alpha_g\beta_g e^{\beta_g x_3}], \quad T_2(\alpha_g, \beta_g) = \alpha_g[\Phi(g)e^{\alpha_g x_3} - 2g^2 e^{\beta_g x}\\
&\gamma_1(\alpha, \beta) = 4\varsigma^2\Phi(\varsigma)e^{\alpha x_3} - F(\varsigma)e^{-\beta x_3} - F_1(\varsigma)e^{\beta x_3}\\
&\gamma_2(\alpha, \beta) = F_1(\varsigma)e^{\alpha x_3} - F(\varsigma)e^{-\alpha x_3} - 4\Phi(\varsigma)\alpha\beta e^{\beta x_3}\\
&\Phi(\varsigma) = 2\varsigma^2 - k^2, \quad \alpha_g = (g^2 - h^2)^{1/2}, \quad \beta_g = (g^2 - k^2)^{1/2}, \quad g = \kappa, p\\
&F(\varsigma) = \Phi^2(\varsigma) - 4\varsigma^2\alpha\beta, \quad F_1(\varsigma) = \Phi^2(\varsigma) + 4\varsigma^2\alpha\beta, \quad F_2(\varsigma) = F(\varsigma)F_1(\varsigma)\\
&F'(\varsigma) = 8\varsigma\Phi(\varsigma) - 8\varsigma\alpha\beta - 4\varsigma^3\left(\frac{\beta}{\alpha} + \frac{\alpha}{\beta}\right)
\end{aligned}
\tag{3.22}
$$

In formulae (3.20) and (3.21) we must put

$$Z_3(\varsigma) = \Omega^-(\xi) \text{ when } |x_1| \le l, \quad x_3 \le 0 \tag{3.23}$$

$$P_1^{(n)} = 0, \quad Q_1^{(n)} = 0, \quad Z_3(\varsigma) = \Omega^-(\xi) + \Omega^-(-\xi) \text{ when } x_1 > l, \quad x_3 \le 0 \tag{3.24}$$

The second auxiliary problem for an elastic half-space. The following stresses are specified on the surface of the elastic body $x_3 = 0$:

$$
\begin{aligned}
&\sigma_{33}|_{x_3=0} = 0, \quad \sigma_{13}|_{x_3=0} = X_{13}, \quad X_{13} = \sin px_1, \quad |x_1| \le l\\
&\sigma_{33}|_{x_3=0} = 0, \quad \sigma_{13}|_{x_3=0} = 0, \quad |x_1| > l
\end{aligned}
\tag{3.25}
$$

In exactly the same way as in the previous case, the use of a Fourier transformation leads to the following formulae

$$\sigma_{33}^*|_{x_3=0} = 0, \quad \sigma_{13}^*|_{x_3=0} = X_{13}^*, \quad X_{13}^* = \frac{1}{4\pi}(\Omega^+(\xi) - \Omega^+(-\xi)) \tag{3.26}$$

The function $\Omega^+(\xi)$ is a defined by formula (3.15).

From the conditions in the $x_3 = 0$ plane we obtain

$$(2\xi^2 - k^2)A - 2i\xi\beta B = 0, \quad 2i\xi\alpha A + (2\xi^2 - k^2)B = \frac{X_{13}^*}{\mu} \tag{3.27}$$

The constants A and B are defined as the solution of system (3.27), after which the displacements are found by applying an inverse Fourier transformation to expressions (3.11)

$$
\begin{aligned}
u_1 &= \frac{1}{\mu}\int_{-\infty}^{+\infty} (-2\xi^2 e^{\alpha x_3} + (2\xi^2 - k^2)e^{\beta x_3})\frac{\beta X_{13}^*}{F(\xi)} e^{i\xi x_1} d\xi\\
u_3 &= \frac{i}{\mu}\int_{-\infty}^{+\infty} (2\alpha\beta e^{\alpha x_3} - (2\xi^2 - k^2)e^{\beta x_3})\frac{\xi X_{13}^*}{F(\xi)} e^{i\xi x_1} d\xi
\end{aligned}
\tag{3.28}
$$

Integrals (3.28) can be converted to the following form, more convenient for calculations,

$$
\begin{aligned}
u_1^{(n)} &= -\frac{i}{2\mu}H_2^{(n)}e^{-i\kappa x_1} - \frac{i}{2\mu}P_2^{(n)}e^{-ipx_1} - \frac{i}{2}J_{k1}^{(n)} - 2iJ_{h2}^{(n)}\\
u_3^{(n)} &= \frac{1}{2\mu}K_2^{(n)}e^{-i\kappa x_1} + \frac{1}{2\mu}Q_2^{(n)}e^{-ipx_1} + J_{k3}^{(n)} + J_{h4}^{(n)}
\end{aligned}
\tag{3.29}
$$

Here

$$J_{gm}^{(n)} = \frac{1}{\pi\mu} e^{-igx_1} \int_0^\infty S_m^{(n)} e^{-\eta x_1} d\eta, \quad g = k, h, \quad m = 1, 2, 3, 4$$

$$\frac{H_2^{(n)}}{T_2(\beta_\kappa, \alpha_\kappa)} = \frac{K_2^{(n)}}{T_1(\beta_\kappa, \alpha_\kappa)} = \frac{Z_1(-\kappa)}{F'(\kappa)}, \quad \frac{P_2^{(n)}}{T_2(\beta_p, \alpha_p)} = \frac{Q_2^{(n)}}{T_1(\beta_p, \alpha_p)} = \frac{1}{F(p)} \tag{3.30}$$

$$S_1^{(n)} = \gamma_1(\alpha, \beta) \frac{\beta \Phi(\varsigma)}{2F_2(\varsigma)} Z_1(\varsigma) \bigg|_{\varsigma = -k + i\eta}, \quad S_2^{(n)} = \gamma_2(\alpha, \beta) \frac{\varsigma^2 \beta}{4F_2(\varsigma)} Z_1(\varsigma) \bigg|_{\varsigma = -h + i\eta}$$

$$S_3^{(n)} = \gamma_2(\beta, \alpha) \frac{\varsigma \Phi(\varsigma)}{4F_2(\varsigma)} Z_1(\varsigma) \bigg|_{\varsigma = -k + i\eta}, \quad S_4^{(n)} = \gamma_1(\beta, \alpha) \frac{\varsigma \alpha \beta}{2F_2(\varsigma)} Z_1(\varsigma) \bigg|_{\varsigma = -h + i\eta}$$

We have used the notation of (3.22).

In formulae (3.29) and (3.30) we must put

$$Z_1(\varsigma) = \Omega^+(\xi) \text{ when } |x_1| \le l, \quad x_3 \le 0 \tag{3.31}$$

$$P_2^{(n)} = 0, \quad Q_2^{(n)} = 0, \quad Z_1(\varsigma) = \Omega^+(\xi) - \Omega^+(-\xi) \text{ when } x_1 > l, \quad x_3 \le 0 \tag{3.32}$$

Thus, we have obtained solution describing the propagation of waves in the positive direction of the x_1 axis. Formulae for waves propagating in the opposite direction can be written by analogy.

4. THE CONDITIONS FOR CONTACT BETWEEN THE PIEZOELECTRIC STRIP AND THE ELASTIC HALF-SPACE

We will assume that the conditions for ideal contact between the piezoelectric strip and the elastic body in the contact area $|x_1| \le l, x_3 = 0$ are satisfied; they can be written in the form

$$\overset{a}{u}_1 = \overset{s}{u}_1, \quad \overset{a}{u}_3 = \overset{s}{u}_3, \quad \overset{a}{\sigma}_{13} = \overset{s}{\sigma}_{13}, \quad \overset{a}{\sigma}_{33} = \overset{s}{\sigma}_{33} \tag{4.1}$$

The superscripts a or s show that the quantity belongs to the piezoelectric strip or to the elastic half-space respectively.

We will consider the solution of the problem for an elastic half-space in the contact area.

Waves propagate in opposite directions from any point of the contact area $|x_1| \le l, x_3 = 0$. The solutions obtained for the auxiliary problems describe waves propagating in the positive x_1 direction. In order for the contact conditions to be satisfied, we must bear in mind the displacements and stresses of the waves propagating in the negative direction. With this aim in mind the formulae for the displacements in the contact area for each value of $n = 1, 2, \ldots, N$ will be written taking into account the evenness of the functions $u_3^{(n)}(x_1)$ and the oddness of the functions $u_1^{(n)}(x_1)$

$$u_1^{(n)}(x_1) = u_{1+}^{(n)}(+x_1) - u_{1-}^{(n)}(-x_1), \quad u_3^{(n)}(x_1) = u_{3+}^{(n)}(+x_1) + u_{3-}^{(n)}(-x_1) \tag{4.2}$$

The subscripts plus or minus indicate that a displacement belongs to a wave propagating in the positive or negative direction of the x_1 axis respectively.

We will assume that the calculation of the auxiliary problems (Section 3) has been carried out. This means that displacements of the elastic body $u_{1+}^{(n)}$ and $u_{3+}^{(n)}$ have been calculated in the section $|x_1| \le l$ for any n from 1 to N. The formulae for waves propagating in the negative direction of the x_1 axis, are written by analogy with formulae (3.20)–(3.24) and (3.29)–(3.32). The combination of the displacements $u_{1+}^{(n)}, u_{3+}^{(n)}$ and $u_{1-}^{(n)}, u_{3-}^{(n)}$ according to formulae (4.2), gives the total displacements in the contact area. The total displacements calculated in the interval $|x_1 \le l$ can then be extended and expanded in Fourier series in the same way as was done for E_3 (formulae (2.11) and (2.12)). The Fourier series for the total displacements are represented in the following form (the summation over n is carried out from $n = 1$ to N, and the summation over j is carried out from $j = 1$ to N):

for the first auxiliary problem

$$u_1^{(n)} = \sum_j p_{jn}\sin a_j x_1, \quad u_3^{(n)} = \sum_j q_{jn}\cos a_j x_1 \tag{4.3}$$

for the second auxiliary problem

$$u_1^{(n)} = \sum_j y_{jn}\sin a_j x_1, \quad u_3^{(n)} = \sum_j z_{jn}\cos a_j x_1 \tag{4.4}$$

The complete solution of the problem for an elastic half-space is equal to the sum of the solutions of the auxiliary problems, multiplied by the unknown constants A_n and B_n,

$$\begin{gathered} u_1^s = \sum_n\sum_j (A_n p_{jn} + B_n y_{jn})\sin a_j x_1, \quad u_3^s = \sum_n\sum_j (A_n q_{jn} + B_n z_{jn})\cos a_j x_1 \\ \sigma_{33}^s = \sum_n A_1 \cos a_n x_1, \quad \sigma_{13}^s = \sum_n B_n \sin a_n x_1 \end{gathered} \tag{4.5}$$

The constant A_n and B_n are found from the conditions for the piezoelectric strip to be in contact with elastic half-space.

We must then convert the formulae for the stresses of the piezoelectric strip in the contact area.

In the contact area ($|x_1| \le l$) the displacements of the elastic body are equal to the displacements of the piezoelectric strip. According to the second hypothesis (Section 2), the displacements of the piezoelectric strip do not change over the thickness, and hence the formulae for displacements of the elastic body in the contact area (4.5) will henceforth be used to determine the stresses of the piezoelectric strip.

The stress σ_{11}^a can be expressed in the terms of the displacement by substituting expressions (4.5) into relation (2.4) and then into (2.2), taking formulae (2.12) into account. We obtain

$$\sigma_{11}^a = \frac{1}{s_{11}^E(1-\nu_p^2)}\sum_j\left[a_j\sum_n(A_n p_{jn} + B_n y_{jn}) - d_{31}(1+\nu_p)\phi_j\right]\cos a_j x_1 \tag{4.6}$$

The formulae for the stresses σ_{13}^a (2.9) and σ_{33}^a (2.10) can be written in terms of the displacement using Eqs (4.6)

$$\sigma_{13}^a = \sum_j\left[\sum_n \chi_j(A_n p_{jn} + B_n y_{jn}) + \varphi_j\phi_j\right]\sin a_j x_1$$

$$\sigma_{33}^a = \sum_n\left[\sum_n\left(\frac{t}{2}\chi_j a_j(A_n p_{jn} + B_n y_{jn}) + t\rho\omega^2(A_n q_{jn} + B_n z_{jn})\right) + \theta_j\phi_j\right]\cos a_j x_1 \tag{4.7}$$

$$\chi_j = t\left(\rho_p\omega^2 - \frac{a_j^2}{s_{11}^E(1-\nu_p^2)}\right), \quad \varphi_j = \frac{t d_{31} a_j}{s_{11}^E(1-\nu_p)}\phi_j, \quad \theta_j = \frac{t^2 d_{31} a_j^2}{2 s_{11}^E(1-\nu_p)}\phi_j$$

Equations (4.7) describe the stresses σ_{13}^a and σ_{33}^a of the piezoelectric strip in the region where it is in contact with elastic half-space. Equations (4.5) give the same stress of the elastic half-space. According to the last two contact conditions of (4.1) they must be equal.

The system of equations in the constants $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ is obtained by substituting expressions (4.7) and (4.5) into the last two conditions of (4.1) and equating coefficients of the like trigonometric functions. They contain $2N$ equations and $2N$ unknown constants $A_1, \ldots, A_N$, and $B_1, \ldots, B_N$. After reduction, this system reduces to the following more convenient form

$$\sum_n (g_{jn}A_n + b_{jn}B_n) = \varphi_j, \quad \sum_n (c_{jn}A_n + r_{jn}B_n) = \theta_j \tag{4.8}$$

Here

$$g_{jn} = \chi_j p_{jn}, \quad b_{jn} = \chi_j y_{jn} + \delta_{jn}, \quad \delta_{jn} = \begin{cases} 0, & j \neq n \\ 1, & j = n \end{cases}$$
$$c_{jn} = -\frac{ta_j}{2}\chi_j p_{jn} - t\rho\omega^2 q_{jn} + \delta_{jn}, \quad r_{jn} = -\frac{ta_j}{2}\chi_j y_{jn} - t\rho\omega^2 z_{jn} \tag{4.9}$$

After calculating the constants $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ all the required quantities can easily be obtained from the previous formulae: the displacements and stresses in the region $|x_1| \leq l, x_3 = 0$ can be calculated from formulae (4.5) and when $x_3 < 0$ they can be calculated as the sum of the solutions of the first and second auxiliary problems, multiplied by the constants $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$.

It should be noted that the solution obtained for the piezoelectric strip and the elastic half-space holds everywhere with the exception of a small neighbourhood of the points $x_1 = \pm l, x_3 = 0$, where the solution has a singularity. These singular stress–strain states are localized in a small neighbourhood of the singular points and attenuate very rapidly with distance from them. Methods which enable the singular solutions to be obtained are well known (see, for example, [15, 16]).

5. NUMERICAL EXAMPLE

As an example we will consider an elastic half-space of steel, oscillations of which are excited by a piezoelectric strip of PZT-5 piezoelectric ceramics of thickness $t = 0.001$ m and width $2l = 0.02$ m.

The electric load E_3 in the contact area is given by formula (2.7). As noted above, the quantity E_3 can be extended in different ways. Here it is taken in the form of a periodic function with period $2L = 8l$

$$E_3 = \frac{2V}{t}\begin{cases} -1, & 0 \leq x_1 \leq l \\ -((x_1/l - 1)^2 - 1), & l < x_1 \leq 2l \\ ((x_1/l - 3)^2 - 1), & 2l < x_1 \leq 3l \\ 1, & 3l < x_1 \leq 4l \end{cases} \tag{5.1}$$

The Fourier series for function (5.1) has the form

$$E_3 = -\frac{2V}{t}\sum_{n=1}^{N} \phi_{2n-1}\cos a_{2n-1}x_1, \quad \phi_{2n-1} = \frac{(-1)^{n+1}128}{(2n-1)^3\pi^3}\left(1 - \cos\frac{(2n-1)\pi}{4}\right), \quad a_{2n-1} = \frac{(2n-1)\pi}{L}$$

The displacements $u_{1+}^{(n)}$ and $u_{3+}^{(n)}$ of the waves propagating in the positive direction of the x_1 axis are calculated from formulae which give solutions of the first and second auxiliary problems. The total displacements in the contact area $u_1^{(n)}$ and $u_3^{(n)}$ are found from formulae (4.2) as a linear combination of the displacements $u_{1+}^{(n)}, u_{3+}^{(n)}, u_{1-}^{(n)}$ and $u_{3-}^{(n)}$ of waves propagating in the positive and negative directions of the x_1 axis. The total displacements can be extended as follows:

$$u_1^{(n)} = \begin{cases} u_1^{(n)}(x_1), & 0 \leq x_1 \leq l \\ u_1^{(n)}(l), & l \leq x_1 \leq L - 1 \\ u_1^{(n)}(L - x_1), & L - l \leq x_1 \leq L \end{cases}, \quad u_3^{(n)} = \begin{cases} u_3^{(n)}(x_1), & 0 \leq x_1 \leq l \\ (2 - x_1/l)u_3^{(n)}(l), & l \leq x_1 \leq L - l \\ u_3^{(n)}(L - x_1), & L - l \leq x_1 \leq L \end{cases} \tag{5.2}$$

It should be noted that the method of continuation of the functions (5.2) must be chosen so as to ensure rapid convergence of the solution in the form of Fourier series.

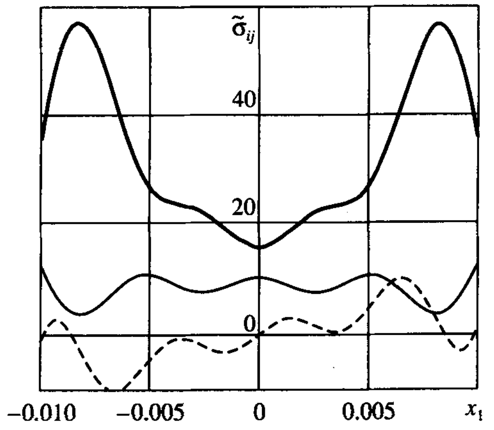

Fig. 2

After this the following steps are carried out: the coefficients p_{jn}, q_{jn}, y_{jn}, z_{jn} of Fourier series (4.3) and (4.4) are calculated. They are then substituted into system (4.8) from which we then find the unknown constants $A_1, \dots, A_N$ and $B_1, \dots, B_N$. In the special case considered the constants A_n and B_n are as follows:

$$\begin{aligned} A_1 &= 0.917\times10^{12}, \quad A_3 = -0.691\times10^{12}, \quad A_5 = 0.427\times10^{12}, \quad A_7 = -0.217\times10^{11} \\ B_1 &= 0.920\times10^{13}, \quad B_3 = -0.578\times10^{13}, \quad B_5 = 0.226\times10^{13}, \quad B_7 = -0.929\times10^{11} \end{aligned} \tag{5.3}$$

It can be seen that the solution obtained converges and the coefficients A_n are considerably less than B_n. This means that the shear load in the contact area plays the main role in exciting acoustic waves, while a normal load of the same value produces displacements and stresses an order of magnitude less.

The results of a calculation are shown in the form of graphs in Figs 2–6 for quantities with a tilde, which are defined by the following formulae

$$\tilde{\sigma}_{1i} = -\frac{\sigma_{i3}s_{11}^E}{Vd_{31}}, \quad \tilde{\sigma}_{11} = -\frac{\sigma_{11}s_{11}^E}{Vd_{31}}, \quad \tilde{u}_i = -\frac{u_i}{Vd_{31}} \tag{5.4}$$

In Fig. 2 we show the distribution of the amplitude values of the stresses $\tilde{\sigma}_{13}$ (the dashed curve), $\tilde{\sigma}_{33}$ (the thin curve) $\tilde{\sigma}_{11}$ and (the thick curve) in the contact area $|x_1| \le l$. In Section 2 we assumed the hypothesis that the stresses σ_{11} are considerably greater than the stresses σ_{33}. The hypothesis is only used in the equations of state. The results of a calculation of the stresses from the equilibrium equations confirm this hypothesis.

In Fig. 3 we show how the displacements of the longitudinal bulk waves (a) and the shear bulk waves (b) attenuate with distance from the source of oscillations on the body surface along the x_1 axis, where the thick curves the displacement $\tilde{u}_3$ and the thin curves show the displacements $\tilde{u}_1$.

The graphs in Fig. 4 show how the amplitude values of the displacements $\tilde{u}_3$ (a) and $\tilde{u}_1$ (b), which are the sum of the displacements of the surface and bulk waves, depend on the x_1 coordinate outside the piezoelectric strip. The thick curves show the displacements on the surface of the elastic body ($x_3 = 0$), and the thin curves show the displacements at a distance of $x_3 = 0.06$ m from the body surface. It can be seen from Fig. 4 that at a distance of greater than 0.1 m from the piezoelectric strip along the surface of the elastic body the waves become almost periodic functions, i.e. the bulk waves attenuate and only the surface waves remain.

Figure 5 shows the attenuation of the maximum values of the displacements of the surface waves $\tilde{u}_3$ (the thick curve) and $\tilde{u}_1$ (the thin curve) deep into the body.

In Fig. 6 the maximum values of the displacements $\tilde{u}_3$ of the surface waves are represented as a function of the oscillation frequency of the piezoelectric strip. It can be seen that for the same intensity of the electric load the displacements depend considerably on the oscillation frequency. There are several maxima in the frequency range investigated. One of them occurs at an angular frequency of the oscillations of 78 kHz. Graphs 2–5 were drawn for this frequency.

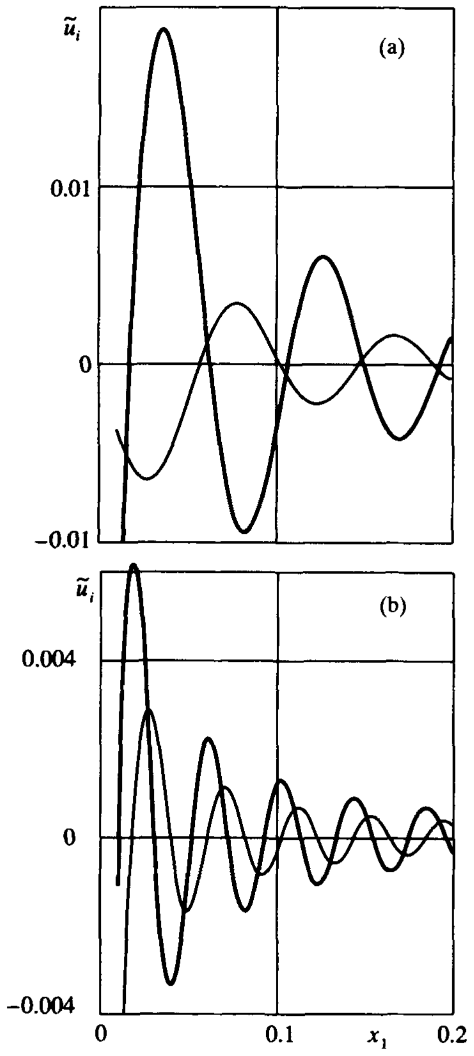

Fig. 3

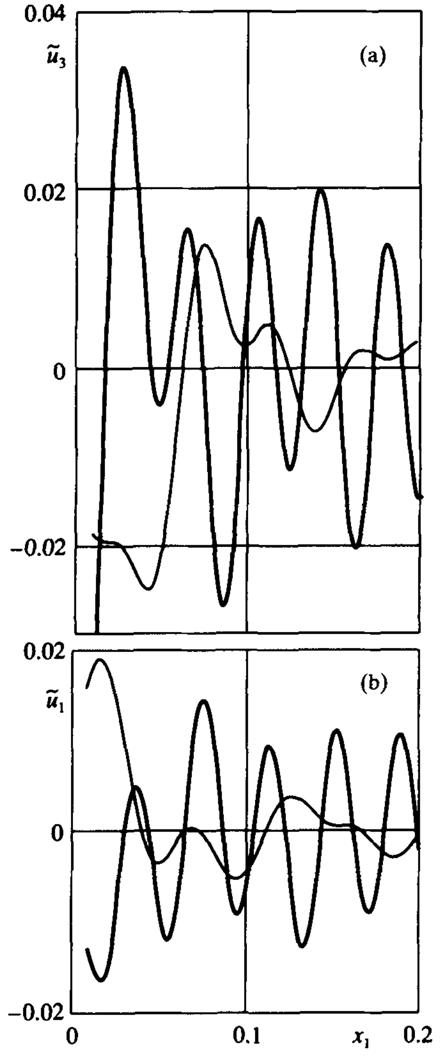

Fig. 4

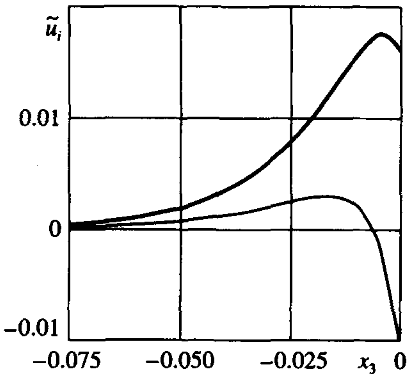

Fig. 5

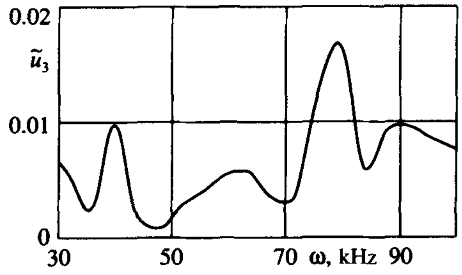

Fig. 6

6. CONCLUSION

We have proposed a method of constructing an analytical solution of the dynamic contact problem for a piezoelectric strip and an elastic half-space. The analytical solution obtained enables all the required quantities to be found, and enables the electroelastic state of piezoelectric strip and the propagation of acoustic waves in an elastic half-space to be investigated.

The proposed method of solution was demonstrated by a numerical example, and as a result of the calculation the displacements and stresses of the piezoelectric strip and of the elastic half-space were obtained both on the body surface and at internal points of the body; the maximum amplitude values of the displacements of the surface waves as a function of the oscillation frequency of the piezoelectric strip were analysed. It was shown that these displacements reach a maximum at a definite frequency; the distance from the source of oscillations at which body waves attenuate and only surface waves remain has been estimated and the attenuation of the waves with distance from the source of oscillations deep inside the body has been investigated.

REFERENCES

1. LORD RAYLEIGH, *Scientific Papers*, Vols 1–6. Cambridge University Press, Cambridge, 1899–1920.
2. LAMB, H., *Dynamic Theory of Sound*, L. E. Arnold, 1931.
3. LOVE, A. E. H., *Some Problems of Geodynamics*. Cambridge University Press, Cambridge, 1911.
4. LOVE, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*. Cambridge University Press, Cambridge, 1944.
5. VIKTOROV, I. A., *Acoustic Surface Waves in Solids*. Nauka, Moscow, 1981.
6. DIEULESANT, E., ROYER, D., *Ondes Elastique dans les Solides*. Masson, Paris, 1974.
7. PARTON, V. Z. and KUDRAYAVTSEV, B. A., *Electromagnetoelasticity of Piezoelectric and Electrically Conducting Solids*. Nauka, Moscow, 1998.
8. WANG, Q., QUEK, S. Y. and VARADAN, V. K., Love waves in piezoelectric coupled solid media. *Smart Mater. Struct*. 2001, **10**, 380–388.
9. BATH, M., *Mathematical Aspects of Seismology*. Elsevier, Amsterdam, 1968.
10. BERLINCOURT, D. A., CURRAN, D. R. and JAFFE, H., Piezoelectric and piezomagnetic materials and their function in transducers. In *Physical Acoustics*, Vol. 1, Part A. (Edited by W. P. Mason). Academic Press, New York, 1964, 169–270.
11. LEE, C. K., Theory of laminated piezoelectric plates for the design of distributed sensors/actuators. P. I. Governing equations and reciprocal relationships. *J. Acoust. Soc. America*, 1990, **87**, 3, 1144–1158.
12. TZOU, H. S., *Piezoelectric Shells (Distributed Sensing and Control of Continua)*. Kluwer, Dordrecht, 1993.
13. ROGACHEVA, N, N., *The Theory of Piezoelectric Shells and Plates*. CRC Press, Boca Raton, FL, 1994.
14. SHENDEROV, Ye. L., *Wave Problems of Hydroacoustic*. Sudostroyeniye, Leningrad, 1972.
15. AKSENTYAN, O. K., Features of the stress–strain state of a plate in the neighbourhoods of a rib. *Prikl. Mat. Mekh*., 1967, **31**, 1, 178–186.
16. WILLIAMS, M. L., The complex-variable approach to stress singularity. *Trans. ASME J. Appl. Mech.*, 1956, **23**, 3, 477–478.

Translated by R.C.G.